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B.Sc - III

MATHEMATCS HONS.: Paper - V
group B. (Multiple integrals).

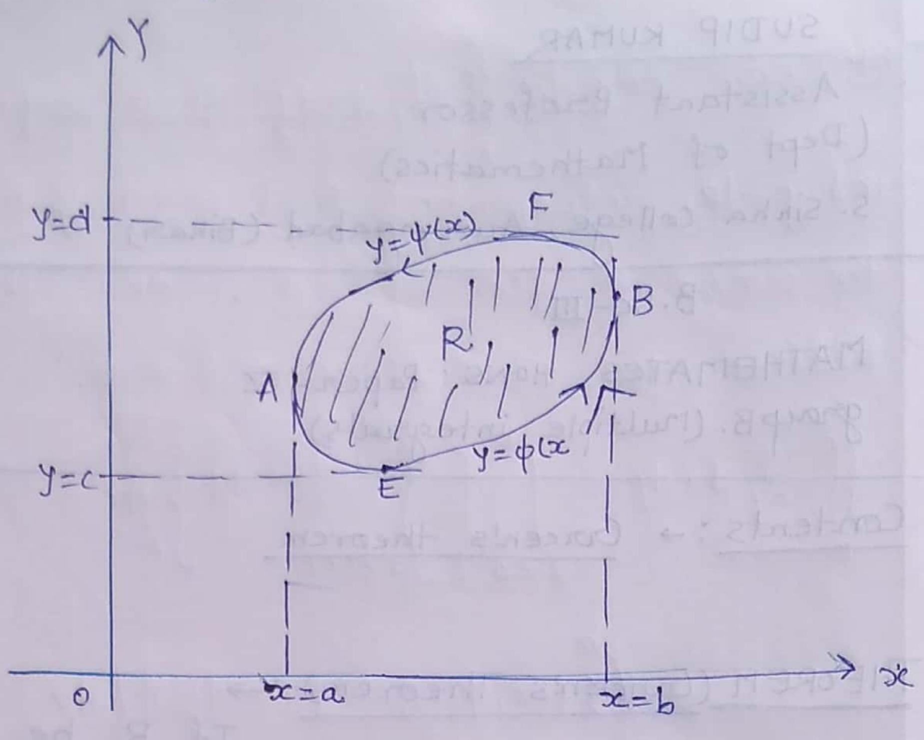
Contents :→ Green's theorem

THEOREM (Green's Theorem) :→ If R be a closed region of the xy-plane bounded by a simple closed regular curve Γ and if P & Q are continuous functions of x and y having continuous partial derivatives in R, then

$$\int_{\Gamma} P\,dx + Q\,dy = \iint_R \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \right) dx\,dy$$

Where the line integral is taken in the positive sense.

Proof:→ Let R be a region bounded by a single closed regular curve Γ which ~~meets~~ is cut in at most two points by any line parallel to the axes. This means that R can be represented by $a \le x \le b$, $\phi(x) \le y \le \psi(x)$ or $c \le y \le d$, $f(y) \le x \le g(y)$.



The equation of the curves AEB and BFA are $y=\phi(x)$ & $y=\psi(x)$ respectively.

The equations of the curves FAE and EBF are however $x=f(y)$ & $x=g(y)$, respectively.

$$\therefore \iint_R \frac{\partial Q}{\partial x}\,dx\,dy = \int_{y=c}^{y=d}\left\{\int_{x=f(y)}^{g(y)} \frac{\partial Q}{\partial x}\,dx\right\}dy$$

$$= \int_{y=c}^{d}\left\{Q(g(y),y) - Q(f(y),y)\right\}dy$$

$$= \int_{y=c}^{d} Q(g(y),y)\,dy + \int_{y=d}^{c} Q(f(y),y)\,dy$$

Where the integrals are taken over arcs $\widehat{EBF}$ & $\widehat{FAE}$, respectively; hence

$$\iint_R \frac{\partial Q}{\partial x}\,dx = \int_\Gamma Q(x,y)\,dy \quad \text{———} \quad (1)$$

We now compute $\iint_R \frac{\partial P}{\partial y}\,dx\,dy$ as a repeated integrals, integrating first w.r.to y between the curves $y = \phi(x)$ & $y = \phi(x)$ and then w.r.to x between the parallels $x = a$ & $x = b$.

$$\iint_R \frac{\partial P}{\partial y}\,dx\,dy = \int_{x=a}^{b} \left\{ \int_{y=\phi(x)}^{\psi(x)} \frac{\partial P}{\partial y}\,dy \right\} dx$$

$$= \int_{x=a}^{b} \left\{ P(x, \psi(x)) - P(x, \phi(x)) \right\} dx$$

$$= - \int_{x=a}^{b} P(x, \phi(x))\,dx - \int_{x=b}^{x=a} P(x, \psi(x))\,dx$$

Where the integrals are taken over arcs $\widehat{AEB}$ and $\widehat{BFA}$, respectively, hence.

$$- \iint_R \frac{\partial P}{\partial y}\,dx\,dy = \int_\Gamma P(x, y)\,dx \quad \text{---} \quad (2)$$

$\therefore$ From (1) & (2), we have

$$\boxed{\iint_R \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \right) dx\,dy = \int_\Gamma P\,dx + Q\,dy}$$

proved.

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